第三章
拉普拉斯變換

習題 3-1

1. 求下列各函數的拉氏變換.

(2) $f(t)=\begin{cases} 5,\ 0\leqslant t<3 \\ 0,\ t\geqslant 3 \end{cases}$

(5) $f(t)=\sin(at+b)$，其中 a 與 b 皆爲常數.

解：(2) $\mathcal{L}\{f(t)\}=\displaystyle\int_0^\infty e^{-st}f(t)\,dt=5\int_0^3 e^{-st}\,dt=-\frac{5e^{-st}}{s}\Bigg|_0^3=\frac{5(1-e^{-3s})}{s}$

(5) $\mathcal{L}\{\sin(at+b)\}=\displaystyle\lim_{h\to\infty}\int_0^h e^{-st}\sin(at+b)\,dt$

$$=\lim_{h\to\infty}\left[\frac{-se^{-st}\sin(at+b)-ae^{-st}\cos(at+b)}{s^2+a^2}\Bigg|_0^h\right]$$

$$=\lim_{h\to\infty}\left[\frac{-se^{-sh}\sin(ah+b)-ae^{-sh}\cos(ah+b)}{s^2+a^2}+\frac{s\sin b+a\cos b}{s^2+a^2}\right]$$

$$=\frac{s\sin b+a\cos b}{s^2+a^2},\quad s>0$$

4. (1) 證明 $\mathcal{L}\{t^\alpha\}=\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$，$\alpha>-1$，$s>0$.

(2) 求 $\mathcal{L}\{t^{-1/2}\}$.

解：(1) $\mathcal{L}\{t^\alpha\}=\displaystyle\int_0^\infty e^{-st}t^\alpha\,dt=\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha d\left(\frac{u}{s}\right)$ (令 $u=st$)

$$=\frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

(2) $\mathcal{L}\{t^{-1/2}\}=\dfrac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}=\dfrac{\sqrt{\pi}}{s^{1/2}}=\sqrt{\dfrac{\pi}{s}}$，$s>0$

5. 試求下列反拉氏變換.

(1) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}$ (4) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}$

解：(1) 因 $\mathcal{L}\{\cos 2t\}=\dfrac{s}{s^2+4}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}=\cos 2t$$

(4) 因 $\mathcal{L}\{te^{-t}\}=\dfrac{1}{(s+1)^2}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2}\right\}=te^{-t}$$

習題 3-2

1. 求下列各函數的拉氏變換.

(1) $2t^2-3t+4$ (4) 10^t+2e^{-t}

(8) $\sin t\cos t$ (9) $\cos t\cos 2t$

解：(1) $\mathcal{L}\{2t^2-3t+4\}=2\mathcal{L}\{t^2\}-3\mathcal{L}\{t\}+4\mathcal{L}\{1\}=\dfrac{4}{s^3}-\dfrac{3}{s^2}+\dfrac{4}{s}$

(4) $\mathcal{L}\{10^t+2e^{-t}\}=\mathcal{L}\{10^t\}+2\mathcal{L}\{e^{-t}\}=\mathcal{L}\{e^{t\ln 10}\}+2\mathcal{L}\{e^{-t}\}$

$$=\frac{1}{s-\ln 10}+\frac{2}{s+1}$$

(8) $\mathcal{L}\{\sin t\cos t\}=\mathcal{L}\left\{\dfrac{1}{2}\sin 2t\right\}=\dfrac{1}{2}\mathcal{L}\{\sin 2t\}=\dfrac{1}{s^2+4}$

(9) $\mathcal{L}\{\cos t\cos 2t\}=\mathcal{L}\left\{\frac{1}{2}(\cos 3t+\cos t)\right\}=\frac{1}{2}\left(\frac{s}{s^2+9}+\frac{s}{s^2+1}\right)$

$$=\frac{s(s^2+5)}{(s^2+9)(s^2+1)}$$

2. 求下列各式的反拉氏變換.

(1) $\frac{6}{2s-3}$　　　　(3) $\frac{2s-18}{s^2+9}$

解：(1) $\mathcal{L}^{-1}\left\{\frac{6}{2s-3}\right\}=3\mathcal{L}^{-1}\left\{\frac{1}{s-3/2}\right\}=3e^{3t/2}$

(3) $\mathcal{L}^{-1}\left\{\frac{2s-18}{s^2+9}\right\}=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-6\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}=2\cos 3t-6\sin 3t$

習題 3-3

求下列各式的反拉氏變換.

1. $\frac{1}{s^2-6s+5}$

解：$\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+5}\right\}=\mathcal{L}^{-1}\left\{\frac{1/4}{s-5}-\frac{1/4}{s-1}\right\}=\frac{1}{4}(e^{5t}-e^{t})$

3. $\frac{s+1}{s(s^2+4)}$

解：$\frac{s+1}{s(s^2+4)}=\frac{1/4}{s}+\frac{-s/4}{s^2+4}+\frac{1}{s^2+4}$

故 $\mathcal{L}^{-1}\left\{\frac{s+1}{s(s^2+4)}\right\}=\frac{1}{4}-\frac{1}{4}\cos 2t+\frac{1}{2}\sin 2t$

8. $\frac{s^2-2}{s(s^2-4)}$

解：$\frac{s^2-2}{s(s^2-4)}=\frac{1/2}{s}+\frac{1/4}{s-2}+\frac{1/4}{s+2}$

故 $\mathcal{L}^{-1}\left\{\dfrac{s^2-2}{s(s^2-4)}\right\}=\dfrac{1}{2}+\dfrac{1}{4}e^{2t}+\dfrac{1}{4}e^{-2t}$

習題 3-4

1. 求下列函數的拉氏變換**.**

(1) $te^{-t}\cos t$　　(3) $t^2\sin at$　　(8) $\dfrac{1-e^{-t}}{t}$

解：(1) $\mathcal{L}\{t\cos t\}=-\dfrac{d}{ds}\left(\dfrac{s}{s^2+1}\right)=\dfrac{s^2-1}{(s^2+1)^2}$

故 $\mathcal{L}\{te^{-t}\cos t\}=\dfrac{(s+1)^2-1}{[(s+1)^2+1]^2}=\dfrac{s(s+2)}{(s^2+2s+2)^2}$

(3) $\mathcal{L}\{t^2\sin at\}=\dfrac{d^2}{ds^2}\left(\dfrac{a}{s^2+a^2}\right)=\dfrac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

(8) $\mathcal{L}\left\{\dfrac{1-e^{-t}}{t}\right\}=\displaystyle\int_s^\infty\left(\frac{1}{u}-\frac{1}{u+1}\right)du=\lim_{h\to\infty}\int_s^h\left(\frac{1}{u}-\frac{1}{u+1}\right)du$

$$=\lim_{h\to\infty}\left[\left(\ln|u|-\ln|u+1|\right)\Big|_s^h\right]$$

$$=\lim_{h\to\infty}\left(\ln\left|\frac{h}{h+1}\right|-\ln\left|\frac{s}{s+1}\right|\right)$$

$$=-\ln\left|\frac{s}{s+1}\right|=\ln\left|\frac{s+1}{s}\right|$$

2. 求下列各式的反拉氏變換**.**

(1) $\dfrac{1}{(s-2)^2}$　　(2) $\dfrac{s^3}{(s^2+1)^2}$

解：(1) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2}\right\}=te^{2t}$

(2) $\mathcal{L}^{-1}\left\{\dfrac{s^3}{(s^2+1)^2}\right\}=\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}-\dfrac{s}{(s^2+1)^2}\right\}=\cos t-\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

$$=\cos t-\frac{1}{2}\,\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$=\cos t-\frac{1}{2}\,t\sin t$$

習題 3-5

解下列各題.

1. $y'+y=\sin x$，$y(0)=0$

解：$\mathcal{L}\{y'\}+\mathcal{L}\{y\}=\mathcal{L}\{\sin x\}$，可得 $(sY(s)-0)+Y(s)=\dfrac{1}{s^2+1}$

$$Y(s)=\frac{1}{(s+1)(s^2+1)}=\frac{1}{2}\left(\frac{1}{s+1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right)$$

故 $$y=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{2}\,(e^{-x}-\cos x+\sin x)$$

3. $\dfrac{dI}{dt}+50I=5$，$I(0)=0$

解：$\mathcal{L}\left\{\dfrac{dI}{dt}\right\}+50\,\mathcal{L}\{I\}=\mathcal{L}\{5\}$

$$s\mathcal{L}\{I\}-0+50\,\mathcal{L}\{I\}=\frac{5}{s}，\mathcal{L}\{I\}=\frac{5}{s(s+50)}=\frac{0.1}{s}-\frac{0.1}{s+50}$$

故 $$I=0.1-0.1e^{-50t}$$

5. $y''+y=2\cos t$，$y(0)=1$，$y'(0)=0$

解：$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=2\mathcal{L}\{\cos t\}$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{2s}{s^2+1}$$

$$Y(s)=\frac{s^3+3s}{(s^2+1)^2}=\frac{s}{s^2+1}+\frac{2s}{(s^2+1)^2}$$

故 $$y=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$=\cos t+\mathcal{L}^{-1}\left\{(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$=\cos t+t\sin t$$

10. $\begin{cases} x'(t)=-x+y \\ y'(t)=-2x-4y \end{cases}$，$x(0)=1$，$y(0)=0$

解：$\begin{cases} sX(s)-1=-X(s)+Y(s) \\ sY(s)=-2X(s)-4Y(s) \end{cases}$

解得 $$X(s)=\frac{s+4}{s^2+5s+6}=\frac{2}{s+2}-\frac{1}{s+3}$$

$$Y(s)=\frac{-2}{s^2+5s+6}=\frac{2}{s+3}-\frac{2}{s+2}$$

故 $$x=2e^{-2t}-e^{-3t},\ y=2(e^{-3t}-e^{-2t})$$

習題 3-6

1. 求下列各函數的拉氏變換**.**

(1) $f(t)=\begin{cases} 1, & 0\leqslant t\leqslant 2 \\ 0, & t>2 \end{cases}$

解：$f(t)=u(t)-u(t-2)$

故 $\mathcal{L}\{f(t)\}=\mathcal{L}\{u(t)\}-\mathcal{L}\{u(t-2)\}=\frac{1}{s}-\frac{e^{-2s}}{s}=\frac{1-e^{-2s}}{s}$**.**

(3) $f(t)=\begin{cases} \cos t, & 0\leqslant t\leqslant \pi \\ 0, & t>\pi \end{cases}$

解：$f(t)=[u(t)-u(t-\pi)]\cos t$

$$\mathcal{L}\{f(t)\}=\mathcal{L}\{u(t)\cos t\}-\mathcal{L}\{u(t-\pi)\cos t\}$$

$$=\mathcal{L}\{u(t)\cos t\}+\mathcal{L}\{u(t-\pi)\cos(t-\pi)\}$$

$$=\frac{s}{s^2+1}+e^{-\pi s}\cdot\frac{s}{s^2+1}-\frac{s(1+e^{-\pi s})}{s^2+1}$$

(5) $u(t-2)t^2$

解：令 $f(t)=t^2$，則 $f'(t)=2t$，$f''(t)=2$，$f'''(t)=0$，

$f(t)$ 在 $t=2$ 處的二次泰勒多項式爲

$$f(t)=f(2)+f'(2)(t-2)+\frac{f''(2)}{2!}(t-2)^2=4+4(t-2)+(t-2)^2$$

故 $\mathcal{L}\{u(t-2)t^2\}=\mathcal{L}\{u(t-2)[4+4(t-2)+(t-2)^2]\}$

$$=e^{-2s}\left(\frac{4}{s}+\frac{4}{s^2}+\frac{2}{s^3}\right)=\frac{2e^{-2s}}{s}\left(2+\frac{2}{s}+\frac{1}{s^2}\right)$$

(6) $u(t-1)(2t^3+3t-2)$

解：令 $f(t)=2t^3+3t-2$，則 $f'(t)=6t^2+3$，$f''(t)=12t$，$f'''(t)=12$，$f^{(4)}(t)=0$，

$f(t)$ 在 $t=1$ 處的三次泰勒多項式爲

$$f(t)=f(1)+f'(1)(t-1)+\frac{f''(1)}{2!}(t-1)^2+\frac{f'''(1)}{3!}(t-1)^3$$
$$=3+9(t-1)+6(t-1)^2+2(t-1)^3$$

故 $\mathcal{L}\{u(t-1)(2t^3+3t-2)\}$

$$=\mathcal{L}\{u(t-1)[3+9(t-1)+6(t-1)^2+2(t-1)^3]\}$$

$$=e^{-s}\left(\frac{3}{s}+\frac{9}{s^2}+\frac{12}{s^3}+\frac{12}{s^4}\right)$$

$$=\frac{3e^{-s}}{s}\left(1+\frac{3}{s}+\frac{4}{s^2}+\frac{4}{s^3}\right)$$

2. 求下列各式的反拉氏變換**.**

(2) $\dfrac{e^{-\pi s/3}}{s^2+1}$　　(4) $\dfrac{e^{-3s}}{s^2+6s+10}$

解：(2) $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s/3}}{s^2+1}\right\}=u\left(t-\dfrac{\pi}{3}\right)\sin\left(t-\dfrac{\pi}{3}\right)$

(4) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2+6s+10}\right\}=\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{(s+3)^2+1}\right\}=u(t-3)\,e^{-3(t-3)}\sin(t-3)$

3. 解 $y''+y=\begin{cases}0, & x<1\\ 2, & x\geqslant 1\end{cases}$，$y(0)=y'(0)=0$.

解：原式寫成 $y''+y=2u(x-1)$，$s^2Y(s)+Y(s)=\dfrac{2e^{-s}}{s}$，

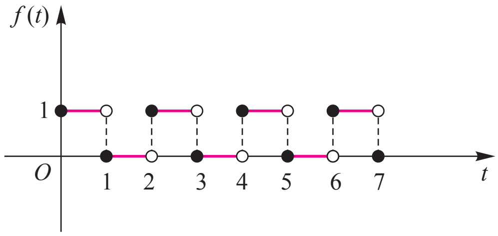
可得 $$Y(s)=\frac{2e^{-s}}{s(s^2+1)}$$

故 $$\begin{aligned} y&=\mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s(s^2+1)}\right\}=2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{se^{-s}}{s^2+1}\right\}\\ &=2u(x-1)-2u(x-1)\cos(x-1)\\ &=2u(x-1)[1-\cos(x-1)]\end{aligned}$$

習題 3-7

求下列各週期函數的拉氏變換.

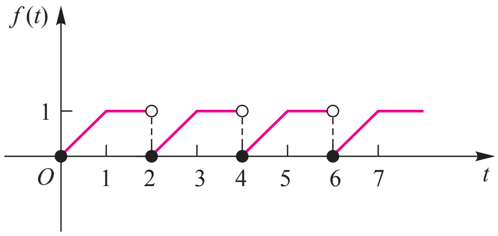
1.



解：函數 $f(t)$ 的週期 $T=2$，$f(t)=\begin{cases}1, & 0\leqslant t<1\\ 0, & 1\leqslant t<2\end{cases}$

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt=\frac{1-e^{-s}}{s(1-e^{-2s})}=\frac{1}{s(1-e^{-s})}$$

3.



解：函數 $f(t)$ 的週期 $T=2$，$f(t)=\begin{cases} t, & 0\leqslant t<1 \\ 1, & 1\leqslant t<2 \end{cases}$

$$\int_0^2 e^{-st} f(t)\,dt=\int_0^1 e^{-st}\,t\,dt+\int_1^2 e^{-st}\,dt=\left(-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right)\Bigg|_0^1+\frac{e^{-st}}{-s}\Bigg|_1^2$$

$$=\frac{1-e^{-s}-se^{-2s}}{s^2}$$

$$\mathcal{L}\{f(t)\}=\frac{\displaystyle\int_0^2 e^{-st} f(t)\,dt}{1-e^{-2s}}=\frac{1-e^{-s}-se^{-2s}}{s^2(1-e^{-2s})}$$

6. $f(t)=|\sin \omega t|$，$t\geqslant 0$

解：函數 $f(t)$ 的週期 $T=\dfrac{\pi}{\omega}$，

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt=\frac{1}{1-e^{-\pi s/\omega}}\left[\frac{\omega(1+e^{-\pi s/\omega})}{s^2+\omega^2}\right]$$

$$=\frac{\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}=\frac{\omega}{s^2+\omega^2}\coth\frac{\pi s}{2\omega}$$

習題 3-8

1. 利用褶積定理求反拉氏變換.

(1) $\dfrac{1}{s^2(s^2+1)}$　　　(3) $\dfrac{s}{(s^2+1)^2}$

解：(1) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\cdot\dfrac{1}{s^2+1}\right\}=t\times\sin t$

$$=\int_0^t (t-\tau)\sin\tau\, d\tau = t\int_0^t \sin\tau\, d\tau - \int_0^t \tau\sin\tau\, d\tau$$

$$=t-t\cos t+t\cos t-\sin t$$

$$=t-\sin t$$

(3) $$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\}=\int_0^t \sin(t-\tau)\cos\tau\, d\tau$$

$$=\int_0^t (\sin t\cos\tau-\cos t\sin\tau)\cos\tau\, d\tau$$

$$=\sin t\int_0^t \cos^2\tau\, d\tau-\cos t\int_0^t \sin\tau\cos\tau\, d\tau$$

$$=(\sin t)\left(\frac{t}{2}+\frac{\sin t\cos t}{2}\right)-(\cos t)\,\frac{\sin^2 t}{2}$$

$$=\frac{1}{2}\,t\sin t$$

解下列各積分方程式.

2. $y(t)=2+\int_0^t y(\tau)\, d\tau$

解：$\mathcal{L}\{y(t)\}=\mathcal{L}\{2\}+\mathcal{L}\left\{\int_0^t 1\cdot y(\tau)\, d\tau\right\}$

$$Y(s)=\frac{2}{s}+\mathcal{L}\{1\times y(t)\}=\frac{2}{s}+\frac{1}{s}\cdot Y(s)$$

可得 $Y(s)=\dfrac{2}{s-1}$，故 $y(t)=2e^t$.

5. $y(t)=\cos t+\int_0^t \sin(t-\tau)\, y(\tau)\, d\tau$

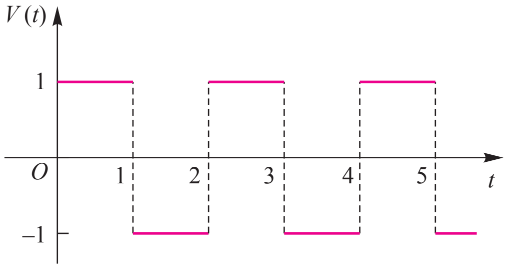
解：$\mathcal{L}\{y(t)\}=\mathcal{L}\{\cos t\}+\mathcal{L}\left\{\int_0^t \sin(t-\tau)\, y(\tau)\, d\tau\right\}$

$$Y(s)=\frac{s}{s^2+1}+\frac{1}{s^2+1}\cdot Y(s)$$

可得 $Y(s)=\frac{1}{s}$，故 $y(t)=1$.

習題 3-9

1. 在例題 3 中，若 $V(t)$ 如圖所示，求 $I(t)$.



解：$L\frac{dI}{dt}+RI=V=u(t)-2u(t-1)+2u(t-2)-2u(t-3)+\cdots$

取拉氏變換可得

$$sL\,\mathcal{L}\{I\}+LI(0)+R\mathcal{L}\{I\}=\frac{1}{s}-\frac{2e^{-s}}{s}+\frac{2e^{-2s}}{s}-\frac{2e^{-3s}}{s}+\cdots$$

以 $I(0)=0$ 代入上式，可得

$$(sL+R)\,\mathcal{L}\{I\}=\frac{1}{s}\,(1-2e^{-s}+2e^{-2s}-2e^{-3s}+\cdots)=\frac{1}{s}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right]$$

$$\mathcal{L}\{I\}=\frac{1}{s(sL+R)}\cdot\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right]$$

$$=\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)+2\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)\sum_{n=1}^{\infty}(-1)^n e^{-ns}$$

故 $I(t)=\left(\frac{1}{R}-\frac{1}{R}e^{-Rt/L}\right)+2\sum_{n=1}^{\infty}(-1)^n u(t-n)\left[\frac{1}{R}-\frac{1}{R}e^{-R(t-n)/L}\right]$

$$=\frac{1}{R}\left[(1-e^{-Rt/L})+2\sum_{n=1}^{\infty}(-1)^n u(t-n)(1-e^{-R(t-n)/L})\right]$$